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$1^{st}$ January, 2018

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Keywords: Implied volatility, Volatility term structure, VIX futures, Realized volatility
JEL classification: C19; C22; C80

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Using an extended LHARG model proposed by Majewski et al. (2015), we derive the closed-form pricing formulas for both the CBOE VIX term structure and VIX futures with different maturities. Our empirical results suggest that the quarterly and yearly components of lagged realized volatility should be added into the model to capture the long-term volatility dynamics. By using the realized volatility based on high frequency data, the proposed model provides superior pricing performance compared with the classic Heston-Nandi GARCH model, both in-sample and out-of-sample. The improvement is more pronounced during high volatility periods.

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## 1 Introduction

The well-known Chicago Board Options Exchange (CBOE) VIX index, computed from a panel of options prices, is a model-free measure of expected average variance for the next 30 days under the risk-neutral measure. The index has become the benchmark for stock market volatility, and is used as the "investor fear gauge" for financial market practitioners. With the launch of VIX futures in 2004 and VIX options in 2006, volatility derivatives have received increasing attention from the market, as the average daily trading volume of VIX futures and VIX options have increased by over 25 and 137 times, respectively, in the last decade. VIX-linked products essentially create a volatility market that enables investors to trade volatility directly, as with equity or fixed income securities[1]. In addition to the VIX index that measures 1-month implied volatility, the CBOE has also launched a series of implied volatility indices across different maturities in recent years, to reflect the volatility term structure under the risk-neutral measure. The CBOE S&P 500 3-month Volatility Index (Ticker:VXV) was launched in November 2007. The CBOE Mid-Term Volatility Index (Ticker: VXMT), a measure of the expected volatility of the S&P 500 index over a 6-month time horizon, was launched in November 2013[2]. The whole family of CBOE VIX indices and VIX futures provides rich information for the implied volatility term structure of the stock market.

Zhang and Zhu (2006) was the first to attempt to price VIX futures based on the classic continuous-time Heston model. The importance of the volatility term structure in VIX futures pricing was illustrated in Zhu and Zhang (2007). Lin (2007) extended the model with simultaneous jumps in both

[1] Luo and Zhang (2014) provide a good discussion of the market for volatility derivatives.
[2] CBOE reported historical data of VXMT back to January 2008.

returns and volatility to price VIX futures with approximation formulas. Adding jumps to the mean-reverting process was also investigated by Sepp (2008), Zhang et al. (2010), and Zhu and Lian (2012), etc. Under the discrete-time Heston-Nandi GARCH framework, Wang et al. (2017) derived the pricing formulas for both VIX and VIX futures. Following the ideas of Hao and Zhang (2013) and Kanniainen et al. (2014), the model parameters were jointly estimated, as the log-likelihoods of both the stock returns (realized volatility if required) and the VIX information were included in the objective function. The empirical results suggested that including the risk-neutral information in the objective function improves parameter estimation and yields better pricing performance.

The seminal work of Andersen et al. (2003) proved the realized volatility computed from high frequency intra-day returns to be an accurate measure of the latent volatility process. Volatility models with the realized measures have attracted great attention in recent years. Leading models include the heterogeneous autoregressive (HAR) model(Corsi (2009)), the MEM model(Engle and Gallo (2006)), the HEAVY model(Shephard and Sheppard (2010)), and the Realized GARCH model(Hansen et al. (2012)). Among these models, the HAR model is receiving increasing attention in volatility modeling and financial applications due to its estimation simplicity and good forecasting performance. The model introduces a cascade structure into the linear autoregression framework, in which the current daily realized variance is regressed on the lagged realized variance over the previous day, week, and month. Empirical studies show that the HAR model provides a parsimonious but effective approximation of the long memory process of volatility[3].

Most studies focus on the performance of the HAR model and its extensions into forecasting volatility or realized volatility under the physical measure, but Corsi et al. (2013) and Majewski et al. (2015) have shown that the HAR framework is also capable of matching the volatility information implied by option prices, i.e., under the risk-neutral measure. Corsi et al. (2013) extended the HAR model with a Gamma innovation (the Heterogeneous Autoregressive Gamma, HARG) and specified an exponentially affine pricing kernel. Majewski et al. (2015) further developed the model by allowing more flexible leverage components (LHARG) and derived the analytical pricing formula for European options. In this study, we extend the LHARG model by including lagged quarterly and yearly realized variance into the RV dynamics, and derive the analytical formulas for the VIX term structure in addition to the VIX futures. We find that adding these two terms enhances the model's ability to capture volatility dynamics over longer horizons, and it is also empirically important for the purpose of pricing VIX term structures and VIX futures. Compared with the pricing formula under the classic Heston-Nandi GARCH model in Wang et al. (2017), our proposed model provides superior performance in pricing VIX term structures and VIX futures. The improvement is more pronounced during high volatility periods when the realized volatility provides more accurate information about underlying volatility than the squared daily returns.

[3] See Corsi et al. (2012) for a review of this model.

A rolling window out-of-sample pricing analysis is also conducted and the main empirical findings are robust.

The remainder of the paper is organized as follows. Section 2 introduces the model setup and derives the pricing formula for VIX term structures and VIX futures. In Section 3, the model estimation using different datasets is discussed. Section 4 presents the empirical results, and Section 5 concludes.

# 2 THE MODEL

## 2.1 LHARG Model and Risk Neutralization

In this study, we denote the original LHARG model of Majewski et al. (2015) as LHARG-M, as it contains volatility components up to the monthly average. We extend the model to LHARG-Q by including the quarterly average (63 trading days), and to LHARG-Y by including both the quarterly and the yearly average (252 trading days). We use LHARG to represent all of these if a property holds for all three models. The LHARG-Y model is specified as

$$R_{t+1} = r + \lambda RV_{t+1} - \frac{1}{2} RV_{t+1} + \sqrt{RV_{t+1}}\epsilon_{t+1}, \quad \epsilon_{t+1} \sim i.i.dN(0,1) \tag{2.1}$$

$$RV_{t+1}|\mathcal{F}_t \sim \Gamma(\delta, \Theta(\boldsymbol{RV_t}, \boldsymbol{L_t}), \theta)$$

$$\begin{aligned}\Theta(\boldsymbol{RV_t}, \boldsymbol{L_t}) = d + &\beta_d RV_t^{(d)} + \beta_w RV_t^{(w)} + \beta_m RV_t^{(m)} + \beta_q RV_t^{(q)} + \beta_y RV_t^{(y)} + \\ &\alpha_d \ell_t^{(d)} + \alpha_w \ell_t^{(w)} + \alpha_m \ell_t^{(m)} + \alpha_q \ell_t^{(q)} + \alpha_y \ell_t^{(y)}\end{aligned}$$

We define the components as follows:

$$\begin{aligned}
RV_t^{(d)} &= RV_t & \ell_t^{(d)} &= \epsilon_t^2 - 1 - 2\gamma\epsilon_t\sqrt{RV_t} \\
RV_t^{(w)} &= \frac{1}{4}\sum_{i=1}^{4} RV_{t-i} & \ell_t^{(w)} &= \frac{1}{4}\sum_{i=1}^{4}(\epsilon_{t-i}^2 - 1 - 2\gamma\epsilon_{t-i}\sqrt{RV_{t-i}}) \\
RV_t^{(m)} &= \frac{1}{17}\sum_{i=5}^{21} RV_{t-i} & \ell_t^{(m)} &= \frac{1}{17}\sum_{i=5}^{21}(\epsilon_{t-i}^2 - 1 - 2\gamma\epsilon_{t-i}\sqrt{RV_{t-i}}) \\
RV_t^{(q)} &= \frac{1}{41}\sum_{i=22}^{62} RV_{t-i} & \ell_t^{(q)} &= \frac{1}{41}\sum_{i=22}^{62}(\epsilon_{t-i}^2 - 1 - 2\gamma\epsilon_{t-i}\sqrt{RV_{t-i}}) \\
RV_t^{(y)} &= \frac{1}{189}\sum_{i=63}^{251} RV_{t-i} & \ell_t^{(y)} &= \frac{1}{189}\sum_{i=63}^{251}(\epsilon_{t-i}^2 - 1 - 2\gamma\epsilon_{t-i}\sqrt{RV_{t-i}})
\end{aligned}$$

$R_t$, $RV_t$, and $r$ denote the log-return of the underlying index, the realized volatility, and the risk-free rate, respectively. $\lambda$ captures the equity risk premium. The conditional distribution of $RV_{t+1}$ features a noncentral gamma distribution (denoted as $\Gamma(\cdot)$) with shape and scale parameters equal to $\delta$ and $\theta$ respectively. The location parameter is given by $\Theta(\boldsymbol{RV_t}, \boldsymbol{L_t})$. Such an autoregressive gamma framework is particularly useful for modeling a non-negative time series, and was thoroughly studied in Gourieroux and Jasiak (2006). Obviously LHARG-Q is nested in LHARG-Y, and the original LHARG-M can be recovered by setting $\beta_q$, $\beta_y$, $\alpha_q$, and $\alpha_y$ to zero.

The specification of the leverage function was inspired by Christoffersen et al. (2008) and enriched by a heterogeneous structure[4]. Unlike the Heston-Nandi leverage function, $\Theta(\boldsymbol{RV_t}, \boldsymbol{L_t})$ is no longer guaranteed to be positive. However, Majewski et al. (2015) provided numerical evidence that the analytical results can effectively describe a regularized version of the model.

According to the properties of the noncentral gamma distribution, the conditional expectation of $RV_{t+1}$ in the physical ($P$) measure is

$$\begin{aligned}\mathbb{E}_t^P[RV_{t+1}] &= \theta\delta + \theta\Theta(\boldsymbol{RV_t}, \boldsymbol{L_t}) \\ &= \theta\delta + \theta d + \theta\beta_d RV_t^{(d)} + \theta\beta_w RV_t^{(w)} + \theta\beta_m RV_t^{(m)} + \theta\beta_q RV_t^{(q)} + \theta\beta_y RV_t^{(y)} + \\ &\quad \theta\alpha_d \ell_t^{(d)} + \theta\alpha_w \ell_t^{(w)} + \theta\alpha_m \ell_t^{(m)} + \theta\alpha_q \ell_t^{(q)} + \theta\alpha_y \ell_t^{(y)}\end{aligned}$$

Therefore, the stationary condition for the LHARG-Y model is

$$\pi^P = \theta(\beta_d + \beta_w + \beta_m + \beta_q + \beta_y) < 1$$

The unconditional expectation of $RV$ under the P-measure is given by

$$\mathbb{E}^P[RV_t] = \frac{\theta\delta + \theta d}{1 - \theta(\beta_d + \beta_w + \beta_m + \beta_q + \beta_y)}$$

When $\theta\delta + \theta d$ is positive and the stationary condition is satisfied, the unconditional variance will be positive.

We can rewrite the formula $\Theta(\boldsymbol{RV_t}, \boldsymbol{L_t})$ as

$$\Theta(\boldsymbol{RV_t}, \boldsymbol{L_t}) = d + \sum_{i=1}^{p} \beta_i RV_{t+1-i} + \sum_{j=1}^{q} \alpha_j \ell_{t+1-j}$$

[4]Majewski et al. (2015) denoted this leverage function as "Zero-Mean" and found it to have the best option pricing performance, because its less-constrained leverage allows the process to explain a larger fraction of the skewness and kurtosis observed in real data.

where

$$\ell_{t+1-j} = \epsilon^2_{t+1-j} - 1 - 2\gamma\epsilon_{t+1-j}\sqrt{RV_{t+1-j}}$$

$$\boldsymbol{RV_t} \equiv (RV_t, RV_{t-1}, ..., RV_{t+1-p}) \qquad \boldsymbol{L_t} \equiv (\ell_t, \ell_{t-1}, ..., \ell_{t+1-q})$$

$$\beta_i = \begin{cases} \beta_d & i = 1 \\ \beta_w/4 & 2 \leq i \leq 5 \\ \beta_m/17 & 6 \leq i \leq 22 \\ \beta_q/41 & 23 \leq i \leq 63 \\ \beta_y/189 & 64 \leq i \leq 252 \end{cases} \qquad \alpha_j = \begin{cases} \alpha_d & j = 1 \\ \alpha_w/4 & 2 \leq j \leq 5 \\ \alpha_m/17 & 6 \leq j \leq 22 \\ \alpha_q/41 & 23 \leq j \leq 63 \\ \alpha_y/189 & 64 \leq j \leq 252 \end{cases}$$

Following Majewski et al. (2015), we specify the following exponentially affine stochastic discount factor,

$$M_{t,t+1} = \frac{\exp(-v_{1t}RV_{t+1} - v_{2t}R_{t+1})}{\mathbb{E}^P_t[\exp(-v_{1t}RV_{t+1} - v_{2t}R_{t+1})]}$$

where $v_{1t}$ is the price of the realized volatility risk and $v_{2t}$ is the price of the equity risk. The non-arbitrage condition requires that $\exp(r) = \mathbb{E}^Q_t(\exp(R_{t+1}))$. With the above stochastic discount factor, we have $\exp(r) = \mathbb{E}^P_t(M_{t+1}\exp(R_{t+1}))$, which implies $v_{2t} = \lambda$. We assume a constant price of realized volatility risk, $v_{1t} \equiv v_1$, so that the structure of the physical dynamics is preserved in the corresponding risk-neutral ($Q$) dynamics[5]:

$$R_{t+1} = r - \frac{1}{2}RV_{t+1} + \sqrt{RV_{t+1}}\epsilon^*_{t+1}, \quad \epsilon^*_{t+1} \sim i.i.dN(0,1)$$

$$RV_{t+1}|\mathcal{F}_t \sim \Gamma(\delta^*, \Theta^*(\boldsymbol{RV_t}, \boldsymbol{L^*_t}), \theta^*)$$

$$\Theta^*(\boldsymbol{RV_t}, \boldsymbol{L^*_t}) = d^* + \sum_{i=1}^{p} \beta^*_i RV_{t+1-i} + \sum_{j=1}^{q} \alpha^*_j \ell^*_{t+1-j}$$

The risk-neutral parameters are linked to the physical parameters through

$$\beta^*_i = [\beta_i + \alpha_i(2\gamma\lambda + \lambda^2)]/(1+\theta y^*), \quad \alpha^*_j = \alpha_j/(1+\theta y^*)$$

$$\delta^* = \delta, \quad d^* = d/(1+\theta y^*), \quad \theta^* = \theta/(1+\theta y^*), \quad \gamma^* = \gamma + \lambda$$

$$\epsilon^*_t = \epsilon_t + \lambda\sqrt{RV_t}$$

$$\ell^*_{t+1-j} = \epsilon^{*2}_{t+1-j} - 1 - 2\gamma^*\epsilon^*_{t+1-j}\sqrt{RV_{t+1-j}}$$

[5]This assumption is commonly cited in the literature, such as in Corsi et al. (2013), Christoffersen et al. (2014).

$$\boldsymbol{L_t^*} \equiv (\ell_t^*, \ell_{t-1}^*, ..., \ell_{t+1-q}^*)$$

where $y^* = (\lambda - 1/2)^2/2 + v_1 - 1/8$. The corresponding stationary and positivity condition under $Q$ are

$$\pi^Q = \theta^*(\beta_d^* + \beta_w^* + \beta_m^* + \beta_q^* + \beta_y^*) < 1 \qquad \theta^*\delta^* + \theta^* d^* > 0$$

## 2.2 The VIX Pricing Formula

In line with Hao and Zhang (2013), the VIX can be calculated as the annualized arithmetic average of the expected daily variance over the following month under the risk-neutral measure:

$$\left(\frac{\mathrm{VIX_t}}{100}\right)^2 = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}_t^Q[RV_{t+k}] \times AF \tag{2.2}$$

where $AF$ is the annualizing factor that converts the daily variance into annualized variance. The implied volatility term structure at time $t$ with the maturity of $n$ is defined as the average expected volatility over the next $n$ trading days:

$$V_t(n) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}_t^Q[RV_{t+k}] \tag{2.3}$$

Assuming that there are 22 trading days in a month and 252 trading days in a year, the model-implied VIX, VXV, and VXMT are[6]:

$$\mathrm{VIX_t} = 100\sqrt{252 V_t(22)} \tag{2.4}$$

$$\mathrm{VXV_t} = 100\sqrt{252 V_t(63)} \tag{2.5}$$

$$\mathrm{VXMT_t} = 100\sqrt{252 V_t(126)} \tag{2.6}$$

**PROPOSITION 1.** *If the return of S&P500 follows the LHARG model, then the model-implied volatility term structure at time t can be presented by a weighted average of the long-run variance $\mu^Q$ and lagged RVs with some zero mean leverage adjustments:*

$$V_t(n) = (1 - \sum_{i=1}^{p} \bar{\xi}_i)\mu^Q + \sum_{i=1}^{p} \bar{\xi}_i RV_{t+1-i} + \sum_{j=1}^{q} \bar{\omega}_j \ell_{t+1-j}^* \tag{2.7}$$

*where*

$$\bar{\xi}_i = \frac{1}{n}\sum_{k=1}^{n} \xi_i^{(k)} \qquad \bar{\omega}_j = \frac{1}{n}\sum_{k=1}^{n} \omega_j^{(k)} \qquad \mu^Q \equiv \mathbb{E}^Q(RV_t) = \frac{\theta^*\delta^* + \theta^* d^*}{1 - \sum_{i=1}^{p} \theta^* \beta_i^*}$$

[6]The selection of the number of trading days is in line with Majewski et al. (2015).

*$\xi_i^{(k)}$ and $\omega_i^{(k)}$ can be obtained by an iterative relationship:*

$$\xi_i^{(k+1)} = \begin{cases} \xi_{i+1}^{(k)} + \xi_1^{(k)}\theta^*\beta_i^* & 1 \le i < p \\ \xi_1^{(k)}\theta^*\beta_i^* & i = p \end{cases} \qquad \omega_j^{(k+1)} = \begin{cases} \omega_{j+1}^{(k)} + \xi_1^{(k)}\theta^*\alpha_j^* & 1 \le j < q \\ \xi_1^{(k)}\theta^*\alpha_j^* & j = q \end{cases}$$

*with initial conditions:* $\xi_i^{(1)} = \theta^*\beta_i^*, \quad \omega_j^{(1)} = \theta^*\alpha_j^*$.

In the LHARG-Y model, the $p$ and $q$ are both equal to 252. The proof of this proposition is given in the Appendix.

Proposition 1 shows that the implied volatility term structure is determined by the weighted average of the past realized volatilities and long-run volatility level with leverage adjustments. Plugging Equation (2.7) into Equations (2.4) - (2.6), we can obtain the VIX, VXV, and VXMT pricing formulas, respectively.

### 2.3 The VIX Futures Pricing Formula

Following the expression given in Proposition 1 of Zhu and Lian (2012), the VIX futures price at time $t$ with maturity time $T$ can be presented as the conditional expectation of VIX under the Q-measure.

$$F(t,T) = \mathbb{E}_t^Q[\mathrm{VIX_T}] = \mathbb{E}_t^Q[100\sqrt{252 V_T(22)}] = \frac{100\sqrt{252}}{2\pi}\int_0^\infty \frac{1-\mathbb{E}_t^Q[\exp(-sV_T(22))]}{s^{3/2}}ds$$

The last term $\mathbb{E}_t^Q[\exp(-sV_T(22))]$ is the moment-generating function of the $V_T(22)$ under the Q-measure.

**PROPOSITION 2.** *Under the risk-neutral measure, the moment-generating function of the $V_T(22)$ has the following form:*

$$f(z,k,\boldsymbol{RV_t},\boldsymbol{L_t^*}) = \mathbb{E}_t^Q[\exp(zV_{t+k}(22))] = \exp\left(\Omega^{(k)} + \sum_{i=1}^{p}\phi_i^{(k)}RV_{t+1-i} + \sum_{j=1}^{q}\Gamma_j^{(k)}\ell_{t+1-j}^*\right)$$

*where $\Omega^{(k)}$, $\phi_i^{(k)}$, and $\Gamma_j^{(k)}$ can be obtained by the following iterations*

$$\Omega^{(k+1)} = \Omega^{(k)} + \mathcal{A}(\phi_1^{(k)},\Gamma_1^{(k)})$$

$$\phi_i^{(k+1)} = \begin{cases} \phi_{i+1}^{(k)} + \mathcal{B}_i(\phi_1^{(k)},\Gamma_1^{(k)}) & 1 \le i < p \\ \mathcal{B}_i(\phi_1^{(k)},\Gamma_1^{(k)}) & i = p \end{cases} \qquad \Gamma_j^{(k+1)} = \begin{cases} \Gamma_{j+1}^{(k)} + \mathcal{C}_j(\phi_1^{(k)},\Gamma_1^{(k)}) & 1 \le j < q \\ \mathcal{C}_j(\phi_1^{(k)},\Gamma_1^{(k)}) & j = q \end{cases}$$

*with the initial values:*

$$\Omega^{(1)} = z\mu^Q(1-\sum_{i=1}^{p}\bar{\xi}_i) + \mathcal{A}(z\bar{\xi}_1, z\bar{\omega}_1)$$

$$\phi_i^{(1)} = \begin{cases} z\bar{\xi}_{i+1} + \mathcal{B}_i(z\bar{\xi}_1, z\bar{\omega}_1) & 1 \leq i < p \\ \mathcal{B}_i(z\bar{\xi}_1, z\bar{\omega}_1) & i = p \end{cases} \qquad \Gamma_j^{(1)} = \begin{cases} z\bar{\omega}_{j+1} + \mathcal{C}_j(z\bar{\xi}_1, z\bar{\omega}_1) & 1 \leq j < q \\ \mathcal{C}_j(z\bar{\xi}_1, z\bar{\omega}_1) & j = q \end{cases}$$

*The definitions of $\mu^Q$, $\bar{\xi}_i$, and $\bar{\omega}_j$ are given in Proposition 1, and $\mathcal{A}(\cdot), \mathcal{B}_i(\cdot), \mathcal{C}_j(\cdot)$ are given as:*

$$\mathcal{A}(\eta, s) = -\frac{1}{2}\log(1-2s) - s - \delta\mathcal{W}(\chi, \theta^*) + d^*\mathcal{V}(\chi, \theta^*)$$

$$\mathcal{B}_i(\eta, s) = \mathcal{V}(\chi, \theta^*)\beta_i^* \qquad \mathcal{C}_j(\eta, s) = \mathcal{V}(\chi, \theta^*)\alpha_j^*$$

$$\mathcal{V}(\chi, \theta^*) = \frac{\theta^*\chi}{1-\theta^*\chi}, \quad \mathcal{W}(\chi, \theta^*) = \log(1-\theta^*\chi), \quad \chi(\eta, s) = \eta + \frac{2s^2\gamma^{*2}}{1-2s}$$

Proof: See the Appendix.

Replace $\mathbb{E}_t^Q[\exp(-sV_T(22))]$ with $f(-s, T-t, \boldsymbol{RV_t}, \boldsymbol{L_t^*})$, and the analytical pricing formula is obtained.

## 2.4 Comparison of the Model

Our model can be compared with the Heston-Nandi GARCH model (HNG) presented by Heston and Nandi (2000). This is a GARCH type model with explicit VIX and VIX futures pricing formulas, which is popular as it is one of the very few discrete-time volatility models that yield analytical solutions for European option prices. The specification of the HNG model under the physical measure is

$$R_{t+1} = r + \lambda h_{t+1} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\epsilon_{t+1}$$

$$h_{t+1} = \omega + \beta h_t + \alpha(\epsilon_t - \gamma\sqrt{h_t})^2$$

The risk-neutral counterpart under LRNVR(Duan (1995)) is

$$R_{t+1} = r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\epsilon_{t+1}^*$$

$$h_{t+1} = \omega + \beta h_t + \alpha(\epsilon_t^* - \gamma^*\sqrt{h_t})^2$$

where $\epsilon_t^* = \epsilon_t + \lambda\sqrt{h_t}$, $\gamma^* = \gamma + \lambda$. The persistence parameters under the $P$ and $Q$ measure are $\pi^P = \beta + \alpha\gamma^2$ and $\pi^Q = \beta + \alpha\gamma^{*2}$, respectively. Pricing formulas for VIX term structures and VIX futures under the HNG can be found in Wang et al. (2017).

# 3 DATA AND ESTIMATION

## 3.1 Data Description

We collect data on S&P 500 index returns and the CBOE panel implied volatility indices including VIX, VXV, and VXMT. We also collect the panel of VIX futures prices with various maturities [7]. The VXMT data start in January 2008, so our full sample spans a 9-year period from January 2008 to March 2017.

To overcome the possible micro-structure noise problem in calculating the realized variance, the $RV$ is calculated using the micro-structure noise robust realized kernel (see Barndorff-Nielsen et al. (2008)).The realized kernel series is trimmed, following Majewski et al. (2015), by removing the most extreme observations (outside the four standard deviation threshold defined by a rolling window of 200 days)[8]. The realized kernel is also re-scaled to match the sample variance of daily close-to-close returns. According to Zhu and Lian (2012), several filters are applied to the VIX futures price data. First, VIX futures with fewer than five days to maturity are removed. Second, data with an associated open interest of fewer than 200 contracts are excluded to avoid any liquidity-related bias. Finally, to match the time horizon in the VIX term structure data, we drop all VIX futures with a time to maturity of over six months. In total, the sample includes 2,270 daily observations for the underlying data and 13,349 observations for VIX futures prices. Table I reports the summary statistics of our data set. For the VIX term structure, we find that 1) the average volatility levels are higher in longer maturities; 2) the standard deviation of the series decreases almost monotonically with maturity, which is reinforced by Figure I, in which the volatility index is smoother for a longer maturity; and 3) all series are skewed and leptokurtic, but the skewness and kurtosis decrease as the maturity increases.

[Insert Table I here]

[Insert Figure 1 here]

## 3.2 Parameter Estimation

Following the ideas of Hao and Zhang (2013), Kanniainen et al. (2014) and Wang et al. (2017), we estimate the LHARG model by maximizing the joint log-likelihood function of the observed market data, i.e., the index return and its realized kernel, VIX term structure (VIX, VXV, and VXMT), and VIX futures prices. We assume independent and normal distributions for the pricing errors associated

[7] The realized measures are collected from the Realized Library at the Oxford-Man institute, and other data can be found on the CBOE's website.

[8] This procedure is proposed by Majewski et al. (2015), and affected about 1.48% of our observations. As a comparison, this procedure affected 1.5% of the observations in Majewski et al. (2015).

with the VIX term structure and VIX futures[9]. The corresponding log-likelihood functions are

- The log-likelihood to match the returns and realized kernels

$$\ell_R = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\Big\{\sum_{t=1}^{T}\log(RV_t) + [R_t - r - \lambda RV_t + \frac{1}{2}RV_t]^2/RV_t\Big\}$$
$$\ell_{RV} = -\sum_{t=1}^{T}\Big(\frac{RV_t}{\theta} + \Theta(\boldsymbol{RV_{t-1}}, \boldsymbol{L_{t-1}})\Big) + \sum_{t=1}^{T}\log\Big(\sum_{k=0}^{\infty}\frac{RV_t^{\delta+k-1}}{\theta^{\delta+k}\Gamma(\delta+k)}\frac{\Theta(\boldsymbol{RV_{t-1}}, \boldsymbol{L_{t-1}})^k}{k!}\Big)$$

- The log-likelihood to match the market implied volatility indices

$$\ell_{\text{VIX}} = -\frac{T}{2}\log(2\pi s^2_{\text{VIX}}) - \frac{1}{2s^2_{\text{VIX}}}\sum_{t=1}^{T}(\text{VIX}_\text{t}^\text{Mod} - \text{VIX}_\text{t}^\text{Mkt})^2$$
$$\ell_{\text{VXV}} = -\frac{T}{2}\log(2\pi s^2_{\text{VXV}}) - \frac{1}{2s^2_{\text{VXV}}}\sum_{t=1}^{T}(\text{VXV}_\text{t}^\text{Mod} - \text{VXV}_\text{t}^\text{Mkt})^2$$
$$\ell_{\text{VXMT}} = -\frac{T}{2}\log(2\pi s^2_{\text{VXMT}}) - \frac{1}{2s^2_{\text{VXMT}}}\sum_{t=1}^{T}(\text{VXMT}_\text{t}^\text{Mod} - \text{VXMT}_\text{t}^\text{Mkt})^2$$

- The log-likelihood to match the market VIX futures prices

$$\ell_{\text{Fut}} = \Big\{-\frac{N}{2}\log(2\pi s^2_{\text{Fut}}) - \frac{1}{2s^2_{\text{Fut}}}\sum_{i=1}^{N}(\text{Fut}_\text{i}^\text{Mod} - \text{Fut}_\text{i}^\text{Mkt})^2\Big\} \times \frac{\text{T}}{\text{N}}$$

where the gamma function $\Gamma(\cdot)$ and the $s^2$ are estimated with the sample variance of pricing errors. Note that the log-likelihood in $\ell_{RV}$ cannot be applied directly because it contains an infinite number of terms. To implement the maximum likelihood estimator, we truncate the infinite sum up to its 90th order, following Majewski et al. (2015). The $T/N$ in $\ell_{\text{Fut}}$ is used to adjust the imbalance in the number of observations between VIX futures prices and other series.

As the variance premium parameter $v_1$ is not identifiable without risk-neutral information in our model, maximizing the joint log-likelihood function $\ell_{R,RV} = \ell_R + \ell_{RV}$ using returns and realized volatility information is not an option. This is not the case for traditional GARCH models, where $\lambda$ determines both equity and the variance premium. The risk-neutral information can be extracted from either the VIX term structure or VIX futures quotes. We therefore use the following estimation methods:

[9] Let $T$ be the number of observations in the returns, realized variance, and volatility indices, and $N$ be the number of VIX futures prices.

- To maximize the joint log-likelihood with the VIX term structure[10]:

$$\ell_{R,RV,\text{VIX}} = \ell_R + \ell_{RV} + \ell_{\text{VIX}} \quad \ell_{R,RV,\text{VXV}} = \ell_R + \ell_{RV} + \ell_{\text{VXV}} \quad \ell_{R,RV,\text{VXMT}} = \ell_R + \ell_{RV} + \ell_{\text{VXMT}}$$

- To maximize the joint log-likelihood with VIX futures:

$$\ell_{R,RV,\text{Fut}} = \ell_R + \ell_{RV} + \ell_{\text{Fut}}$$

For the following empirical analysis, we only keep the daily leverage component and constrain other leverage components to zero to make the model more concise. The main results do not change when additional leverage components are added.

# 4 EMPIRICAL RESULTS

## 4.1 Parameter Estimation

Table II reports the estimated parameters for different LHARG models with different estimation methods. The suffix "-Q" denotes the augmented LHARG with quarterly components, and "-Y" denotes the augmented LHARG with quarterly and yearly components. "VIX1M" (VIX), "VIX3M" (VXV), "VIX6M" (VXMT), and "Fut" (VIX futures) indicate the market data used in the maximum likelihood estimation along with the index returns and realized kernels.

[Insert Table II here]

For the LHARG-M model in columns (1)-(4), the coefficients for all of the lagged volatility components are significantly positive. However, the monthly coefficients increase significantly and the weekly coefficients decrease when VIX3M or VIX6M are used in the estimation. The volatility index is smoother than the realized volatility[11], as it is an average of expected volatility over a certain period. Therefore, the long-term moving average of the realized measures can provide more valuable information than the short-term moving average when a longer-term volatility index is modeled. Similar patterns can be found when VIX futures prices are used in the estimation.

In line with other studies, the persistence parameter $\pi^Q$ is greater than $\pi^P$ and close to 1, which indicates a high persistence in the risk-neutral dynamic. The risk premium parameter $v_1$ is negative in all cases. As shown in Corsi et al. (2013), a negative $v_1$ is consistent with the literature focusing on variance risk premium, such as Bakshi and Kapadia (2003), Carr and Wu (2009). The absolute value of

[10] It is possible to construct a log-likelihood with all three volatility indices simultaneously, but our preliminary results show that the improvement can only be found when compared with the $(R, RV, \text{VIX})$ case. The variance premium parameter $v_1$ may be different for different terms, so we estimate the parameters with each of the three indices one at a time.

[11] See Table I for the standard deviation of Ann.RK and VIXs.

the variance premium parameter decreases when the maturity increases, while the equity risk premium $\lambda$ remains at a stable level. The equity premium and the variance risk premium therefore appear to be quite different, providing additional motivation for separating $v_1$ from $\lambda$.

For the LHARG-Q model in columns (5)-(8), the quarterly coefficient $\theta\beta_q$ is positive and significant across different estimation methods. The quarterly coefficients exceed the monthly coefficients when a longer-term volatility index is fitted. The log-likelihood value of $\ell_{R,RV}$ changes slightly, while larger improvements are found in $\ell_{R,RV,\text{VIX}}$ and $\ell_{R,RV,\text{Fut}}$, indicating the importance of the newly added components. Similar results can be found for the LHARG-Y model in columns (9)-(12). The classic volatility cascade structure is restored where the longer-term volatility component has a smaller coefficient, possibly because LHARG-Y has sufficient explicit volatility information, as the maturity of VIX futures is restricted to a maximum of 180 days.

[Insert Table III here]

Table III reports the results for HNG. In HNG models there is no room for an independent variance risk premium within a single shock, so the equity risk premium $\lambda$ fluctuates violently across different methods. Using risk-neutral information, the model provides a larger equity premium parameter $\lambda$ than its physical counterpart (see the "R" method). The estimated equity risk premium must therefore be inflated to fit the Q information, which is not the case when equity risk and variance risk premium are modeled with separate parameters, as in the LHARG models.

### 4.2 Full-sample Pricing Performance

Table IV reports the volatility indices and the VIX futures pricing performance of different models and methods. Panels A through C are associated with VIX1M, VIX3M, and VIX6M, respectively.

[Insert Table IV here]

For VIX1M and VIX3M, all of the LHARG models outperform the HNG in terms of RMSE. When pricing VIX6M, the HNG model outperforms the LHARG-M model for all estimation methods. This is not the case for our augmented LHARG models. In particular, the LHARG-Y always provides a smaller RMSE than the other two LHARG models and dominates the HNG model. These findings indicate that 1) including realized measures when the VIX term structure or VIX futures are priced is important; and 2) the model structure is also important as a model without explicit long-term information (i.e., the yearly component) can be outperformed by a model without realized measures.

[Insert Table V here]

Table V presents the full-sample RMSE for VIX futures where the model parameters are estimated jointly with the index return, the realized kernel (for LHARG models), and the VIX futures prices. We still find that LHARG-Y is the best model, while HNG dominates LHARG-M and delivers similar results to LHARG-Q. In addition, the total RMSE for VIX futures pricing is decomposed by the VIX level and the time to maturity when the price is quoted. The first dimension is linked to the model's ability to generate an appropriate variance risk premium. The second dimension is linked to the model's ability to track volatility dynamics at different time horizons. Again, the LHARG-Y model performs better across all maturities and extreme volatility periods. The performances of the LHARG-M and HNG models are mixed: the LHARG-M model performs better for short-term VIX futures, while HNG is better for the longer-term. A possible explanation is that the high-frequency data, while providing useful real-time information on latent volatility, are overreacting to short-term shocks in the view of pricing long-term VIX futures. Taking the average of the realized measures over a much longer time span significantly reduces the sensitivity of the model in reacting to temporary shocks.

[Insert Figure II here]

Figure II presents the relationship between the VIX futures pricing error (RMSE) and the maturity in different volatility intervals for the HNG and LHARG-Y models. The models here are also jointly estimated by index return, realized kernels (for LHARG models), and VIX futures prices. The results show that the LHARG-Y model outperforms HNG in almost all respects, particularly during high volatility periods.

### 4.3 Out-of-Sample Pricing Performance

The proposed models are more complex than the Heston-Nandi GARCH model and have several additional parameters, and include the LHARG-M and LHARG-Q model as special cases, so there may be concerns about in-sample overfitting. Thus, we also conduct an out-of-sample pricing analysis based on a rolling window of 252 trading days, with the parameters updated on a monthly basis. We evaluate the out-of-sample pricing errors from 200902-201703 (the observations in 2008 are used as a pre-sample to obtain the first set of parameters) with 2018 volatility indices and 11909 VIX futures prices.

[Insert Table VI here]

Table VI presents the out-of-sample version of Table IV. For the short-term implied volatility indices such as VIX1M and VIX3M, all of the LHARG models still outperform HNG regardless of the estimation method. For a long-term volatility index, HNG only performs slightly better than LHARG-Y (the reduction in the RMSE is less than 1%) when the parameters are estimated with VIX1M, and in the remainder of the cases LHARG-Y dominates.

[Insert Table VII here]

For VIX futures pricing, Table VII shows that in most cases the LHARG-Y model delivers the fewest pricing errors [12], while the results for LHARG-M and HNG are mixed. Thus, most of the empirical

[12] In line with Table V, we only report the results where the parameters are estimated jointly with futures prices. The results for other methods are similar and are available upon request.

results from the previous section remain in the out-of-sample analysis. We do not find significant evidence of in-sample overfitting for our proposed model.

## 5 CONCLUSION

This paper extends the LHARG model proposed in Majewski et al. (2015) by adding quarterly and yearly lagged realized volatility into the HAR structure. The derived analytical pricing formulas for both the CBOE VIX term structure and VIX futures with different maturities are based on the extended model. We estimate the model parameters by jointly matching the information from both the physical and the risk-neutral measures. Our empirical results suggest that the quarterly and yearly components of lagged realized volatility should be added into the model to capture the long-term volatility dynamics. With the realized volatility based on high frequency data, the proposed model provides superior pricing performance to the classic Heston-Nandi GARCH model, both in-sample and out-of-sample. The improvement is more pronounced during high volatility periods.

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## Appendix A: Proof of Proposition 1

Following the definition of the Gamma distribution, we have:

$$\begin{aligned}\mathbb{E}_t^Q[RV_{t+1}] &= \theta^*\delta^* + \theta^*\Theta^*(\boldsymbol{RV_t}, \boldsymbol{L_t^*}) \\ &= \theta^*\delta^* + \theta^* d^* + \sum_{i=1}^{p} \theta^*\beta_i^* RV_{t+1-i} + \sum_{j=1}^{q} \theta^*\alpha_j^*\ell_{t+1-j}^*\end{aligned}$$

Let $\mu^Q \equiv \mathbb{E}^Q(RV_t) = (\theta^*\delta^* + \theta^* d^*)/(1 - \sum_{i=1}^{p}\theta^*\beta_i^*)$, then

$$\mathbb{E}_t^Q[RV_{t+1}] - \mu^Q = \sum_{i=1}^{p} \theta^*\beta_i^*(RV_{t+1-i} - \mu^Q) + \sum_{j=1}^{q} \theta^*\alpha_j^*\ell_{t+1-j}^*$$

Suppose the formula of $\mathbb{E}_t^Q[RV_{t+1}] - \mu^Q$ is

$$\mathbb{E}_t^Q[RV_{t+k}] - \mu^Q = \sum_{i=1}^{p} \xi_i^{(k)}(RV_{t+1-i} - \mu^Q) + \sum_{j=1}^{q} \omega_j^{(k)}\ell_{t+1-j}^*$$

When $k=1$, $\xi_i^{(1)} = \theta^* \beta_i^*, \omega_j^{(1)} = \theta^* \alpha_j^*$. When $k \neq 1$, we have

$$\begin{aligned}
\mathbb{E}_t^Q[RV_{t+k+1}] - \mu^Q &= \mathbb{E}_t^Q[\mathbb{E}_{t+1}^Q[RV_{t+k+1} - \mu^Q]] \\
&= \mathbb{E}_t^Q[\sum_{i=1}^{p} \xi_i^{(k)} (RV_{t+2-i} - \mu^Q) + \sum_{j=1}^{q} \omega_j^{(k)} \ell_{t+2-j}^*] \\
&= \mathbb{E}_t^Q[\xi_1^{(k)} (RV_{t+1} - \mu^Q) + \omega_1^{(k)} \ell_{t+1}^*] + \sum_{i=2}^{p} \xi_i^{(k)} (RV_{t+2-i} - \mu^Q) + \sum_{j=2}^{q} \omega_j^{(k)} \ell_{t+2-j}^* \\
&= \sum_{i=1}^{p} \xi_1^{(k)} \theta^* \beta_i^* (RV_{t+1-i} - \mu^Q) + \sum_{j=1}^{q} \xi_1^{(k)} \theta^* \alpha_j^* \ell_{t+1-j}^* + \sum_{i=1}^{p-1} \xi_{i+1}^{(k)} (RV_{t+1-i} - \mu^Q) + \sum_{j=1}^{q-1} \omega_{j+1}^{(k)} \ell_{t+1-j}^* \\
&= \sum_{i=1}^{p} \xi_i^{(k+1)} (RV_{t+1-i} - \mu^Q) + \sum_{j=1}^{q} \omega_j^{(k+1)} \ell_{t+1-j}^*
\end{aligned}$$

$$\xi_i^{(k+1)} = \begin{cases} \xi_{i+1}^{(k)} + \xi_1^{(k)} \theta^* \beta_i^* & 1 \leq i < p \\ \xi_1^{(k)} \theta^* \beta_i^* & i = p \end{cases} \qquad \omega_j = \begin{cases} \omega_{j+1}^{(k)} + \xi_1^{(k)} \theta^* \alpha_j^* & 1 \leq j < q \\ \xi_1^{(k)} \theta^* \alpha_j^* & j = q \end{cases}$$

So the $V_t(n)$ can be expressed as

$$\begin{aligned}
V_t(n) &= \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_t^Q[RV_{t+k}] \\
&= \sum_{i=1}^{p} \bar{\xi}_i RV_{t+1-i} + \sum_{j=1}^{q} \bar{\omega}_j \ell_{t+1-j}^* + \mu^Q (1 - \sum_{i=1}^{p} \bar{\xi}_i)
\end{aligned}$$

where $\bar{\xi}_i = \frac{1}{n} \sum_{k=1}^{n} \xi_i^{(k)}, \bar{\omega}_j = \frac{1}{n} \sum_{k=1}^{n} \omega_j^{(k)}$.

## Appendix B: Proof of Proposition 2

$$\begin{aligned}
V_t(n) &= \mu^Q (1 - \sum_{i=1}^{p} \bar{\xi}_i) + \sum_{i=1}^{p} \bar{\xi}_i RV_{t+1-i} + \sum_{j=1}^{q} \bar{\omega}_j \ell_{t+1-j}^* \\
&= a^* + \sum_{i=1}^{p} b_i^* RV_{t+1-i} + \sum_{j=1}^{q} c_i^* \ell_{t+1-j}^*
\end{aligned}$$

where $a^* = \mu^Q (1 - \sum_{i=1}^{p} \bar{\xi}_i)$, $b_i^* = \bar{\xi}_i$, $c_j^* = \bar{\omega}_j$. Using the results in Corsi et al. (2013), the moment-generating function of $RV_{t+1}$ is

$$\mathbb{E}_t^Q[\exp(\eta RV_{t+1})] = \exp(\frac{\eta}{1 - \eta \theta^*} \theta^* \Theta^* (\boldsymbol{RV_t}, \boldsymbol{L_t^*}) - \delta \log(1 - \eta \theta^*))$$

Then we have

$$
\begin{aligned}
\mathbb{E}_t^Q[\exp(zRV_{t+1} + s\ell_{t+1}^*)] &= \mathbb{E}_t^Q[\exp(zRV_{t+1} - s - 2s\gamma^*\epsilon_{t+1}^*\sqrt{RV_{t+1}} + s\epsilon_{t+1}^{*2})] \\
&= \mathbb{E}_t^Q[\exp(zRV_{t+1} - s)\mathbb{E}^Q[\exp(-2s\gamma^*\sqrt{RV_{t+1}}\epsilon_{t+1}^* + s\epsilon_{t+1}^{*2})|RV_{t+1}] \\
&= \mathbb{E}_t^Q[\exp(zRV_{t+1} - s - \frac{1}{2}\log(1-2s) + \frac{2s^2\gamma^{*2}RV_{t+1}}{1-2s})] \\
&= \exp(-\frac{1}{2}\log(1-2s) - s - \delta\log(1-\chi\theta^*) + \frac{\chi}{1-\chi\theta^*}\theta^*\Theta^*(\boldsymbol{RV_t}, \boldsymbol{L_t^*})) \\
&= \exp(\mathcal{A}(z,s) + \sum_{i=1}^{p}\mathcal{B}_i(z,s)RV_{t+1-i} + \sum_{j=1}^{q}\mathcal{C}_j(z,s)\ell_{t+1-j}^*)
\end{aligned}
$$

where

$$\mathcal{A}(z,s) = -\frac{1}{2}\log(1-2s) - s - \delta\mathcal{W}(\chi,\theta^*) + d^*\mathcal{V}(\chi,\theta^*)$$

$$\mathcal{B}_i(z,s) = \mathcal{V}(\chi,\theta^*)\beta_i^*$$

$$\mathcal{C}_j(z,s) = \mathcal{V}(\chi,\theta^*)\alpha_j^*$$

$$\mathcal{V}(\chi,\theta^*) = \frac{\theta^*\chi}{1-\theta^*\chi}, \quad \mathcal{W}(\chi,\theta^*) = \log(1-\theta^*\chi), \quad \chi(z,s) = z + \frac{2s^2\gamma^{*2}}{1-2s}$$

Combining the above two equations, $\mathbb{E}_t^Q[\exp(zV_{t+1}(n))]$ is

$$
\begin{aligned}
\mathbb{E}_t^Q[\exp(zV_{t+1}(n))] &= \mathbb{E}_t^Q[\exp(z(a^* + \sum_{i=1}^{p}b_i^*RV_{t+2-i} + \sum_{j=1}^{q}c_j^*\ell_{t+2-j}^*))] \\
&= \mathbb{E}_t^Q[\exp(zb_1^*RV_{t+1} + zc_1^*\ell_{t+1}^*)] \times \exp(za^* + \sum_{i=1}^{p-1}zb_{i+1}^*RV_{t+1-i} + \sum_{j=1}^{q-1}zc_{j+1}^*\ell_{t+1-j}^*) \\
&= \exp(\mathcal{A}(zb_1^*, zc_1^*) + \sum_{i=1}^{p}\mathcal{B}_i(zb_1^*, zc_1^*)RV_{t+1-i} + \sum_{j=1}^{q}\mathcal{C}_j(zb_1^*, zc_1^*)\ell_{t+1-j}^*) \\
&\quad \times \exp(za^* + \sum_{i=1}^{p-1}zb_{i+1}^*RV_{t+1-i} + \sum_{j=1}^{q-1}zc_{i+j}^*\ell_{t+1-j}^*) \\
&= \exp(\Omega^{(1)} + \sum_{i=1}^{p}\phi_i^{(1)}RV_{t+1-i} + \sum_{j=1}^{q}\Gamma_j^{(1)}\ell_{t+1-j}^*)
\end{aligned}
$$

$$\Omega^{(1)} = za^* + \mathcal{A}(zb_1^*, zc_1^*)$$

$$\phi_i^{(1)} = \begin{cases} zb_{i+1}^* + \mathcal{B}_i(zb_1^*, zc_1^*) & 1 \le i < p \\ \mathcal{B}_i(zb_1^*, zc_1^*) & i = p \end{cases} \qquad \Gamma_j^{(1)} = \begin{cases} zc_{j+1}^* + \mathcal{C}_j(zb_1^*, zc_1^*) & 1 \le j < q \\ \mathcal{C}_j(zb_1^*, zc_1^*) & j = q \end{cases}$$

Assume $\mathbb{E}_t^Q[\exp(zV_{t+k}(n))]$ adopts the following formula

$$\mathbb{E}_t^Q[\exp(zV_{t+k}(n))] = \exp(\Omega^{(k)} + \sum_{i=1}^{p} \phi_i^{(k)} RV_{t+1-i} + \sum_{j=1}^{q} \Gamma_j^{(k)} \ell^*_{t+1-j})$$

When $k = 1$, it holds. When $k > 1$, we have

$$\begin{aligned}
\mathbb{E}_t^Q[\exp(zV_{t+k+1}(n))] &= \mathbb{E}_t^Q[\mathbb{E}_{t+1}^Q[\exp(zV_{t+k+1}(n))]] \\
&= \mathbb{E}_t^Q[\exp(\Omega^{(k)} + \sum_{i=1}^{p} \phi_i^{(k)} RV_{t+2-i} + \sum_{j=1}^{q} \Gamma_j^{(k)} \ell^*_{t+2-j})] \\
&= \mathbb{E}_t^Q[\exp(\phi_1^{(k)} RV_{t+1} + \Gamma_1^{(k)} \ell^*_{t+1})] \times \exp(\Omega^{(k)} + \sum_{i=1}^{p-1} \phi_{i+1}^{(k)} RV_{t+1-i} + \sum_{j=1}^{q-1} \Gamma_{j+1}^{(k)} \ell^*_{t+1-j}) \\
&= \exp(\mathcal{A}(\phi_1^{(k)}, \Gamma_1^{(k)}) + \sum_{i=1}^{p} \mathcal{B}_i(\phi_1^{(k)}, \Gamma_1^{(k)}) RV_{t+1-i} + \sum_{j=1}^{q} \mathcal{C}_j(\phi_1^{(k)}, \Gamma_1^{(k)}) \ell^*_{t+1-j}) \times
\end{aligned}$$

$$\begin{aligned}
&\exp(\Omega^{(k)} + \sum_{i=1}^{p-1} \phi_{i+1}^{(k)} RV_{t+1-i} + \sum_{j=1}^{q-1} \Gamma_{j+1}^{(k)} \ell^*_{t+1-j}) \\
=&\exp(\Omega^{(k+1)} + \sum_{i=1}^{p} \phi_i^{(k+1)} RV_{t+1-i} + \sum_{j=1}^{q} \Gamma_j^{(k+1)} \ell^*_{t+1-j})
\end{aligned}$$

$$\Omega^{(k+1)} = \Omega^{(k)} + \mathcal{A}(\phi_1^{(k)}, \Gamma_1^{(k)})$$

$$\phi_i^{(k+1)} = \begin{cases} \phi_{i+1}^{(k)} + \mathcal{B}_i(\phi_1^{(k)}, \Gamma_1^{(k)}) & 1 \le i < p \\ \mathcal{B}_i(\phi_1^{(k)}, \Gamma_1^{(k)}) & i = p \end{cases} \qquad \Gamma_j^{(k+1)} = \begin{cases} \Gamma_{j+1}^{(k)} + \mathcal{C}_j(\phi_1^{(k)}, \Gamma_1^{(k)}) & 1 \le j < q \\ \mathcal{C}_j(\phi_1^{(k)}, \Gamma_1^{(k)}) & j = q \end{cases}$$

**TABLE I**

Descriptive Statistics (2008-2017)

| | N | Mean | Std | Skew | Kurt | Min | Max |
|---|---|---|---|---|---|---|---|
| Panel A: S&P 500 Returns and VIX Term Structure | | | | | | | |
| Return($\times 10^2$) | 2270 | 0.05 | 1.27 | 0.02 | 10.51 | −9.47 | 10.96 |
| Ann.RV | 2270 | 16.27 | 11.84 | 3.00 | 13.60 | 2.96 | 108.65 |
| VIX1M | 2270 | 20.69 | 9.85 | 2.34 | 7.01 | 10.32 | 80.86 |
| VIX3M | 2270 | 22.27 | 8.54 | 1.99 | 4.72 | 12.24 | 69.24 |
| VIX6M | 2270 | 23.58 | 7.50 | 1.68 | 3.16 | 14.04 | 61.47 |
| Panel B: VIX Futures | | | | | | | |
| Total | 13349 | 22.78 | 7.17 | 1.56 | 3.22 | 11.73 | 66.23 |
| Partitioned by VIX Level | | | | | | | |
| VIX≤15 | 4244 | 16.95 | 1.95 | 0.41 | 1.30 | 11.73 | 26.60 |
| 15<VIX≤20 | 4120 | 20.64 | 2.81 | 0.51 | −0.29 | 14.55 | 30.85 |
| 20<VIX≤25 | 2301 | 25.08 | 3.05 | 0.21 | −0.71 | 18.45 | 32.55 |
| 25<VIX≤30 | 1112 | 27.95 | 2.88 | −0.18 | −0.86 | 20.20 | 34.55 |
| 30<VIX≤35 | 523 | 31.45 | 2.58 | −1.41 | 3.06 | 20.08 | 36.85 |
| 35<VIX | 1049 | 39.95 | 6.92 | 0.54 | 3.91 | 17.80 | 66.23 |
| Partitioned by Maturity | | | | | | | |
| DTM≤30 | 2211 | 21.20 | 8.71 | 2.14 | 5.26 | 8.73 | 66.23 |
| 30<DTM≤60 | 2195 | 22.06 | 7.65 | 1.76 | 3.46 | 12.70 | 57.62 |
| 60<DTM≤90 | 2352 | 22.79 | 7.20 | 1.63 | 3.13 | 13.45 | 59.77 |
| 90<DTM≤120 | 2323 | 23.21 | 6.60 | 1.34 | 1.82 | 14.25 | 52.26 |
| 120<DTM≤150 | 2163 | 23.60 | 6.20 | 1.13 | 0.99 | 14.85 | 47.07 |
| 150<DTM | 2105 | 23.88 | 5.91 | 0.98 | 0.52 | 15.20 | 45.26 |

Note: Kurt is excess kurtosis. Ann.RV is the annualized daily volatility calculated by $\sqrt{252 \times Adj.RK} \times 100$ and $Adj.RK = RK \times var(ret)/mean(RK)$. VIX1M, VIX3M, and VIX6M are CBOE VIX, VXV, and VXMT.

**TABLE II**

Parameter Estimation of LHARG Models

| Model | LHARG-M | | | | LHARG-Q | | | | LHARG-Y | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
| Method | VIX1M | VIX3M | VIX6M | Fut | VIX1M | VIX3M | VIX6M | Fut | VIX1M | VIX3M | VIX6M | Fut |
| $\lambda$ | 3.567 | 3.568 | 3.565 | 3.561 | 3.568 | 3.568 | 3.567 | 3.557 | 3.611 | 3.568 | 3.569 | 3.557 |
| | (2.688) | (1.859) | (1.607) | (0.918) | (5.115) | (1.543) | (2.461) | (0.552) | (0.254) | (2.454) | (6.126) | (1.855) |
| $\theta(\times 10^{-5})$ | 3.351 | 3.475 | 3.503 | 3.508 | 3.276 | 3.346 | 3.400 | 3.455 | 3.294 | 3.362 | 3.424 | 3.487 |
| | (0.237) | (0.286) | (0.310) | (0.295) | (0.231) | (0.237) | (0.266) | (0.265) | (0.240) | (0.255) | (0.289) | (0.199) |
| $\delta$ | 1.497 | 1.612 | 1.610 | 1.682 | 1.421 | 1.531 | 1.560 | 1.640 | 1.429 | 1.540 | 1.596 | 1.675 |
| | (0.053) | (0.051) | (0.055) | (0.050) | (0.054) | (0.050) | (0.050) | (0.056) | (0.057) | (0.052) | (0.051) | (0.055) |
| $\theta d(\times 10^{-5})$ | -3.556 | -4.139 | -4.296 | -4.267 | -3.364 | -3.743 | -3.853 | -3.910 | -3.502 | -3.974 | -4.169 | -4.250 |
| | (0.266) | (0.335) | (0.366) | (0.398) | (0.262) | (0.264) | (0.298) | (0.318) | (0.221) | (0.299) | (0.358) | (0.229) |
| $\theta\beta_d$ | 0.457 | 0.474 | 0.470 | 0.479 | 0.437 | 0.422 | 0.432 | 0.456 | 0.435 | 0.410 | 0.420 | 0.443 |
| | (0.027) | (0.035) | (0.038) | (0.035) | (0.028) | (0.029) | (0.033) | (0.033) | (0.030) | (0.032) | (0.035) | (0.032) |
| $\theta\beta_w$ | 0.291 | 0.189 | 0.178 | 0.173 | 0.344 | 0.341 | 0.332 | 0.300 | 0.337 | 0.330 | 0.313 | 0.294 |
| | (0.032) | (0.045) | (0.048) | (0.054) | (0.031) | (0.038) | (0.040) | (0.032) | (0.028) | (0.039) | (0.042) | (0.037) |
| $\theta\beta_m$ | 0.160 | 0.245 | 0.267 | 0.246 | 0.088 | 0.061 | 0.024 | 0.026 | 0.091 | 0.090 | 0.078 | 0.062 |
| | (0.014) | (0.019) | (0.020) | (0.016) | (0.012) | (0.013) | (0.013) | (0.004) | (0.011) | (0.014) | (0.024) | (0.011) |
| $\theta\beta_q$ | | | | | 0.050 | 0.089 | 0.119 | 0.107 | 0.036 | 0.065 | 0.075 | 0.064 |
| | | | | | (0.004) | (0.008) | (0.011) | (0.005) | (0.004) | (0.007) | (0.008) | (0.003) |
| $\theta\beta_y$ | | | | | | | | | 0.025 | 0.028 | 0.031 | 0.036 |
| | | | | | | | | | (0.002) | (0.002) | (0.003) | (0.001) |
| $\theta\alpha_d(\times 10^{-6})$ | 2.513 | 1.900 | 1.554 | 1.869 | 2.547 | 1.651 | 1.398 | 1.602 | 4.100 | 3.261 | 3.071 | 3.460 |
| | (0.321) | (0.444) | (0.475) | (0.614) | (0.306) | (0.365) | (0.365) | (0.272) | (0.455) | (0.366) | (0.508) | (0.434) |
| $\gamma$ | 323.1 | 311.0 | 316.8 | 305.8 | 336.5 | 408.4 | 435.8 | 411.5 | 239.5 | 241.8 | 230.5 | 219.8 |
| | (32.0) | (35.3) | (44.7) | (51.4) | (36.8) | (68.6) | (88.3) | (64.4) | (20.8) | (27.5) | (30.5) | (29.3) |
| $\upsilon_1$ | -1151.2 | -903.7 | -746.6 | -919.1 | -1150.8 | -994.7 | -926.9 | -1084.8 | -1078.8 | -1009.3 | -942.4 | -1120.7 |
| | (486.0) | (419.5) | (563.7) | (365.5) | (563.2) | (410.0) | (473.5) | (521.5) | (161.1) | (385.4) | (561.9) | (208.1) |
| $\pi^P$ | 0.908 | 0.908 | 0.915 | 0.897 | 0.918 | 0.912 | 0.907 | 0.888 | 0.923 | 0.923 | 0.917 | 0.898 |
| $\pi^Q$ | 0.989 | 0.972 | 0.968 | 0.962 | 0.998 | 0.981 | 0.971 | 0.964 | 0.999 | 0.994 | 0.984 | 0.978 |
| **logL** | | | | | | | | | | | | |
| $\ell_{R,RV}$ | 26909 | 26904 | 26897 | 26904 | 26897 | 26905 | 26898 | 26904 | 26904 | 26915 | 26917 | 26922 |
| $\ell_{R,RV,\text{VIX}}$ | **21374** | 21202 | 21070 | 21195 | **21621** | 21420 | 21128 | 21118 | **21813** | 21666 | 21513 | 21444 |
| $\ell_{R,RV,\text{VXV}}$ | 20567 | **20971** | 20902 | 20950 | 21034 | **21445** | 21364 | 21326 | 21494 | **21790** | 21722 | 21540 |
| $\ell_{R,RV,\text{VXMT}}$ | 19613 | 20634 | **20757** | 20635 | 20031 | 21074 | **21208** | 21054 | 20752 | 21473 | **21605** | 21367 |
| $\ell_{R,RV,\text{Fut}}$ | 19687 | 20426 | 20347 | **20458** | 19985 | 20733 | 20722 | **20852** | 20785 | 21088 | 21106 | **21313** |

Note: Here we report $\theta\beta_i$ and $\theta\alpha_i$ instead of $\beta_i$ and $\alpha_i$ to make the results more comparable among different models. Sandwich-type robust standard errors are in parentheses. The second row indicates the information set used. For simplicity, we use "VIX1M" to denote "R+RV+VIX1M", as well as "VIX3M", "VIX6M", and "Fut". The bold number in log-likelihood indicates the largest within each row in each model.

**TABLE III**

Parameter Estimation of Heston-Nandi GARCH Model

| Method | R | R+VIX1M | R+VIX3M | R+VIX6M | R+Fut |
|---|---|---|---|---|---|
| $\lambda$ | 4.098 | 6.981 | 8.202 | 9.020 | 6.105 |
| | (2.894) | (0.611) | (0.558) | (0.590) | (0.291) |
| $\omega(\times 10^{-14})$ | 9.358 | 9.358 | 9.358 | 9.358 | 9.358 |
| | (6.449) | (0.677) | (2.015) | (0.704) | (0.630) |
| $\beta$ | 0.735 | 0.697 | 0.667 | 0.665 | 0.769 |
| | (0.029) | (0.019) | (0.020) | (0.019) | (0.024) |
| $\alpha(\times 10^{-6})$ | 5.556 | 3.275 | 2.057 | 1.596 | 1.364 |
| | (0.969) | (0.212) | (0.059) | (0.045) | (0.062) |
| $\gamma$ | 199.5 | 293.2 | 391.7 | 446.9 | 403.5 |
| | (24.7) | (11.3) | (12.2) | (13.8) | (12.9) |
| $\pi^P$ | 0.957 | 0.979 | 0.983 | 0.984 | 0.991 |
| $\pi^Q$ | 0.966 | 0.992 | 0.996 | 0.997 | 0.998 |
| **logL** | | | | | |
| $\ell_R$ | **7339.7** | 7256.7 | 7228.4 | 7217.2 | 7186.1 |
| $\ell_{R,\text{VIX}}$ | -198.5 | **445.6** | 372.7 | 268.8 | 151.4 |
| $\ell_{R,\text{VXV}}$ | -331.8 | 721.4 | **891.9** | 829.9 | 722.2 |
| $\ell_{R,\text{VXMT}}$ | -431.2 | 615.6 | 1036.9 | **1138.3** | 1076.9 |
| $\ell_{R,\text{Fut}}$ | -377.1 | 570.6 | 974.8 | 1091.3 | **1123.8** |

Note: Sandwich-type robust standard errors are in parentheses. The first row indicates the information set used. For simplicity, we name each column with the information set used for the estimation. For example, "R+Fut" means the parameters are estimated with index returns and VIX futures prices. The bold number in log-likelihood indicates the largest within each row.

**TABLE IV**

Full-Sample Pricing RMSE for VIX Term Structure

| Method | LHARG-M | LHARG-Q | LHARG-Y | HNG |
|---|---|---|---|---|
| | | Panel A: VIX1M Pricing | | |
| R+RV+VIX1M | 2.771 | 2.472 | **2.279** | 4.862 |
| R+RV+VIX3M | 2.982 | 2.711 | **2.443** | 4.959 |
| R+RV+VIX6M | 3.152 | 3.074 | **2.616** | 5.166 |
| | | Panel B: VIX3M Pricing | | |
| R+RV+VIX1M | 3.953 | 3.202 | **2.622** | 4.306 |
| R+RV+VIX3M | 3.303 | 2.681 | **2.314** | 3.945 |
| R+RV+VIX6M | 3.394 | 2.770 | **2.386** | 4.035 |
| | | Panel C: VIX6M Pricing | | |
| R+RV+VIX1M | 6.018 | 4.981 | **3.637** | 4.512 |
| R+RV+VIX3M | 3.830 | 3.157 | **2.660** | 3.701 |
| R+RV+VIX6M | 3.618 | 2.968 | **2.513** | 3.522 |

Note: For the HNG model, the information sets used do not include RV, i.e., "R+RV+VIX1M" for HNG is actually "R+VIX1M". The bold number indicates the minimum RMSE within each row.

**TABLE V**

Full-Sample Pricing RMSE for VIX Futures Pricing

| Model | LHARG-M | LHARG-Q | LHARG-Y | HNG |
|---|---|---|---|---|
| Total RMSE | 4.140 | 3.480 | **2.863** | 3.496 |
| Panel A: Partitioned by VIX Level | | | | |
| VIX≤15 | 3.330 | 2.905 | **2.210** | 2.606 |
| 15<VIX≤20 | 2.763 | 2.443 | **1.765** | 2.466 |
| 20<VIX≤25 | 4.226 | 3.920 | 3.126 | **2.865** |
| 25<VIX≤30 | 4.987 | 4.396 | 3.786 | **3.467** |
| 30<VIX≤35 | 5.261 | 4.330 | **5.103** | 6.507 |
| 35<VIX | 7.996 | 5.747 | **4.762** | 7.235 |
| Panel B: Partitioned by Maturity | | | | |
| DTM≤30 | 3.967 | 3.344 | **2.929** | 4.457 |
| 30<DTM≤60 | 4.129 | 3.330 | **2.699** | 3.447 |
| 60<DTM≤90 | 4.016 | 3.313 | **2.674** | 3.260 |
| 90<DTM≤120 | 4.089 | 3.415 | **2.759** | 3.106 |
| 120<DTM≤150 | 4.280 | 3.658 | **2.967** | 3.205 |
| 150<DTM | 4.370 | 3.824 | **3.152** | 3.345 |

Note: This table only reports the pricing performance for the "Fut" method (i.e., "R+Fut"for HNG and "R+RV+Fut" for LHARG models.). The bold number indicates the minimum RMSE within each row.

**TABLE VI**

Out-of-sample Pricing RMSE for VIX Term Structure

| Method | LHARG-M | LHARG-Q | LHARG-Y | HNG |
|---|---|---|---|---|
| Panel A: VIX1M Pricing | | | | |
| R+RV+VIX1M | 2.116 | 2.039 | **1.912** | 3.722 |
| R+RV+VIX3M | 2.206 | 2.156 | **2.080** | 3.681 |
| R+RV+VIX6M | 2.505 | 2.452 | **2.339** | 3.783 |
| Panel B: VIX3M Pricing | | | | |
| R+RV+VIX1M | 3.966 | 3.504 | **2.574** | 3.935 |
| R+RV+VIX3M | 2.296 | 2.010 | **1.746** | 3.253 |
| R+RV+VIX6M | 2.269 | 2.098 | **1.913** | 3.109 |
| Panel C: VIX6M Pricing | | | | |
| R+RV+VIX1M | 7.839 | 7.618 | 4.667 | **4.652** |
| R+RV+VIX3M | 3.681 | 3.315 | **3.039** | 3.488 |
| R+RV+VIX6M | 2.508 | 2.115 | **1.735** | 2.852 |

Note: The out-of-sample performance evaluation is based on a rolling window of 252 trading days, with the parameters updated on a monthly basis. We evaluate the out-of-sample pricing errors of 2018 trading days (i.e., 200902-201703, the observations in 2008 are used as a pre-sample to get the first parameter for the out-of-sample analysis.). For the HNG model, the information sets used do not include RV, i.e., “R+RV+VIX1M” for HNG is actually “R+VIX1M”. The bold number in each panel indicates the minimum RMSE within each row.

**TABLE VII**

Out-of-sample Pricing RMSE for VIX Futures

| Model | LHARG-M | LHARG-Q | LHARG-Y | HNG |
|---|---|---|---|---|
| Total RMSE | 3.040 | 2.659 | **2.223** | 3.299 |
| Panel A: Partitioned by VIX Level | | | | |
| VIX≤15 | 2.062 | 1.915 | **1.283** | 1.924 |
| 15<VIX≤20 | 2.754 | 2.376 | **1.795** | 2.772 |
| 20<VIX≤25 | 3.137 | 2.601 | **2.432** | 3.879 |
| 25<VIX≤30 | 3.368 | **3.161** | 3.351 | 4.956 |
| 30<VIX≤35 | 4.181 | **3.536** | 4.195 | 5.139 |
| 35<VIX | 6.816 | 5.903 | **4.284** | 6.348 |
| Panel B: Partitioned by Maturity | | | | |
| DTM≤30 | 2.898 | 2.861 | **2.799** | 3.630 |
| 30<DTM≤60 | 2.729 | 2.515 | **2.311** | 3.383 |
| 60<DTM≤90 | 2.729 | 2.334 | **2.004** | 3.258 |
| 90<DTM≤120 | 2.994 | 2.438 | **1.908** | 3.146 |
| 120<DTM≤150 | 3.292 | 2.718 | **1.971** | 3.139 |
| 150<DTM | 3.557 | 3.065 | **2.242** | 3.209 |

Note: This table only reports the out-of-sample pricing performance for the "Fut" method (i.e., "R+Fut"for HNG and "R+RV+Fut" for LHARG models.). The bold number indicates the minimum RMSE within each row.

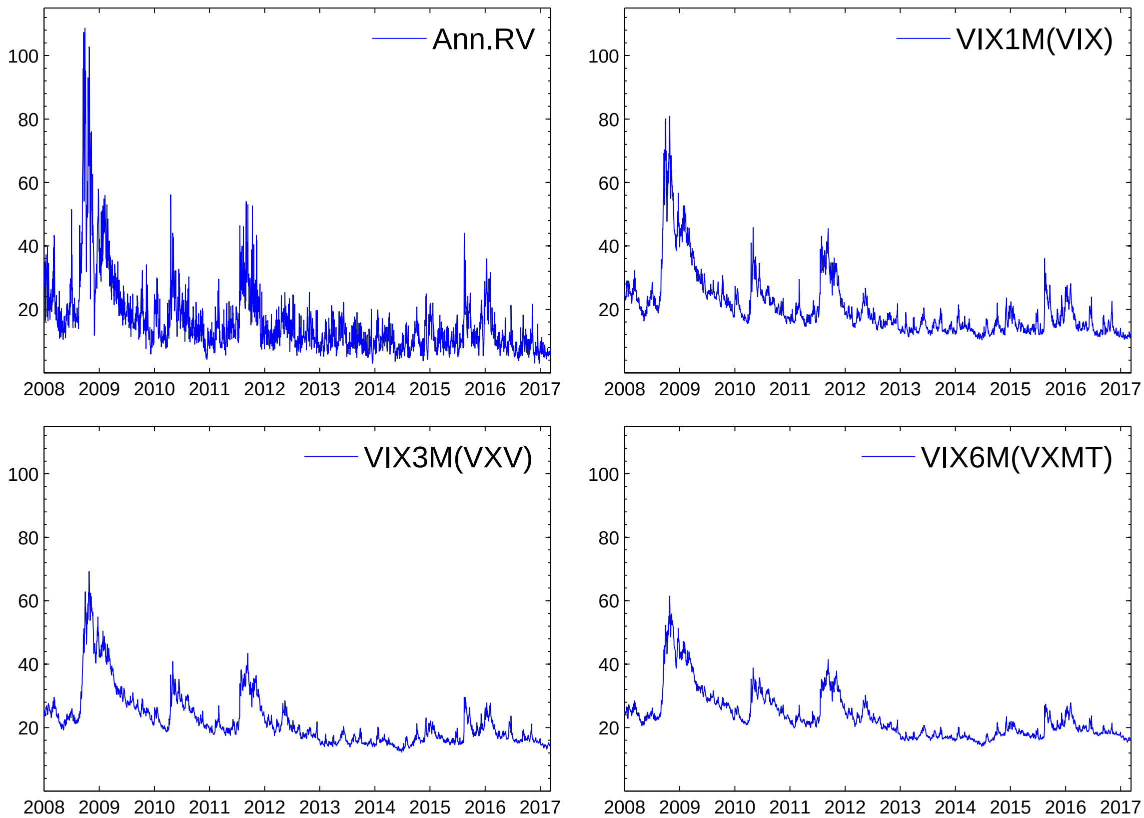


**FIGURE 1**
Time Series Data for Ann.RV, VIX, VXV, and VXMT

Note: Reported are S&P500 annualized realized volatility, and time series data of VIX, VXV, and VXMT.

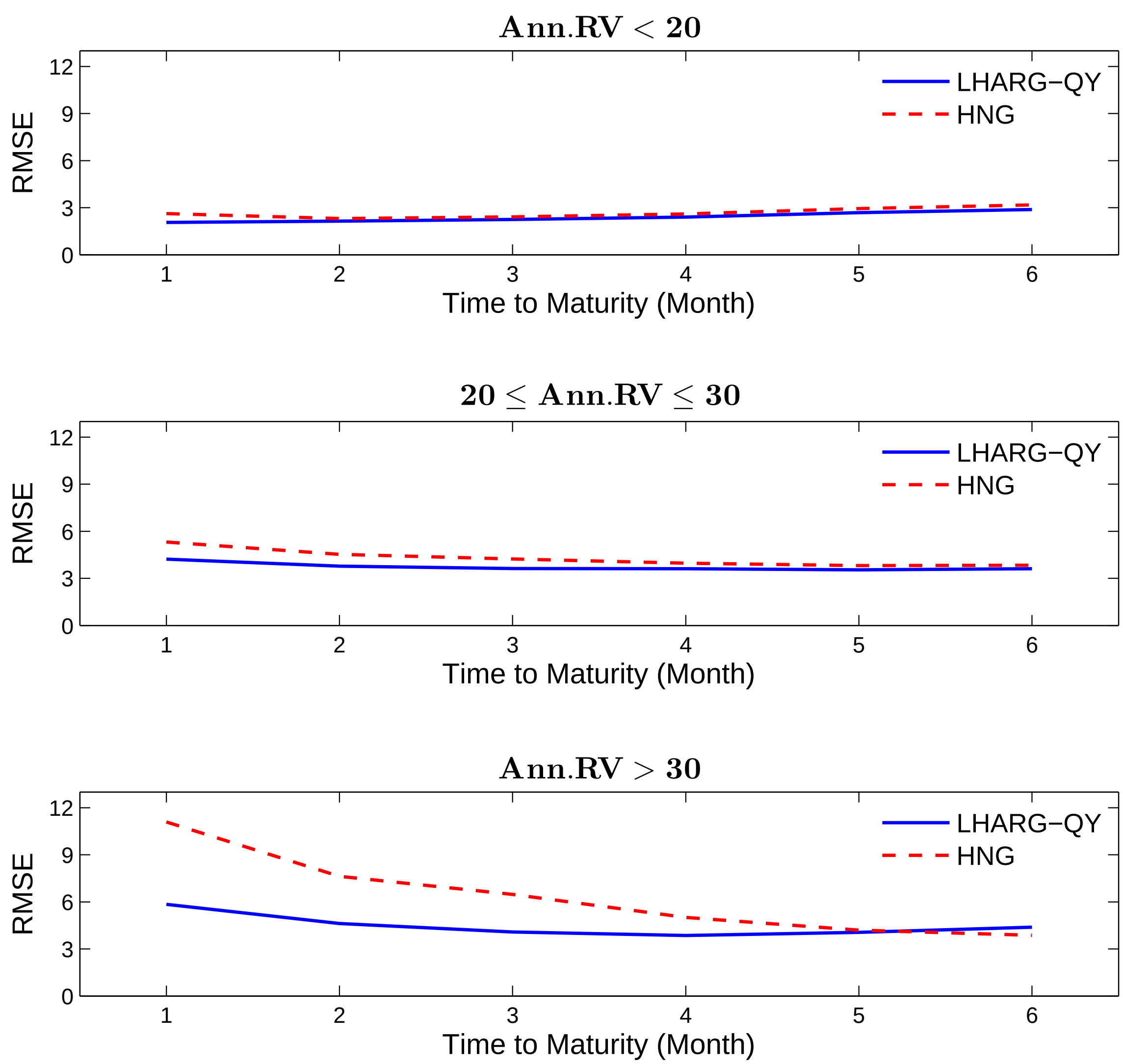


**FIGURE 2**

Full-sample Pricing RMSE for VIX Futures Pricing under Different Volatility Periods

Note: This graph only reports the pricing performance for the "Fut" method (i.e., "Ret+Fut" for HNG and "Ret+RV+Fut" for LHARG-Y). The number X in the horizontal axis denotes the maturity within (30(X-1),30X] days.